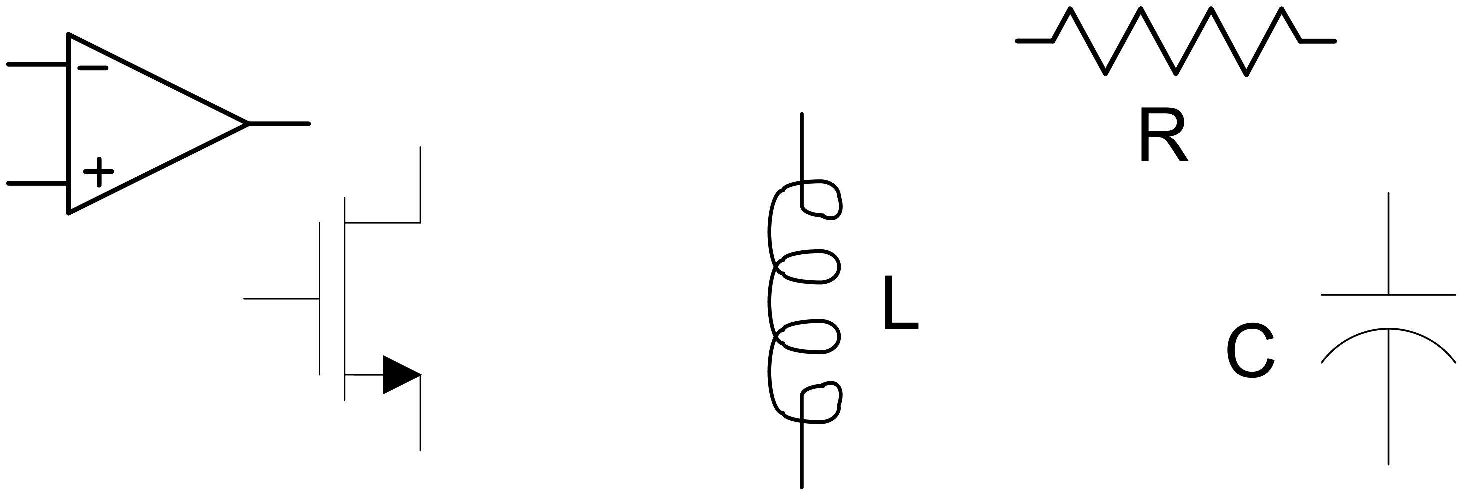
EE 508
Lecture 14

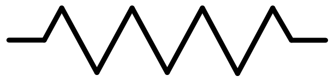
Statistical Characterization of Filter Characteristics

Effects of manufacturing variations on components



- A rigorous statistical analysis can be used to analytically predict how components vary and how component variations impact circuit performance
- Monte Carlo simulations are often used to simulate effects of component variations
  - Requires minimal statistical knowledge to use MC simulations
  - Simulation times may be prohibitively long to get useful results
  - Gives little insight into specific source of problems
  - Must be sure to correctly include correlations in setup
- Often key statistical information is not readily available from the foundry

Modeling process variations in semiconductor processes



R

$$X = X_{NOM} + x_{RPROC} + x_{RWAFER} + x_{RDIE} + x_{RLGRAD} + x_{RLVAR}$$

x_{RPROC}, x_{RWAFER}, x_{RDIE}, x_{RLVAR} often assumed to be Gausian with zero mean

Magnitude of x_{RLGRAD} is usually assumed Gaussian with zero mean, direction is uniform from 0^o to 360^o

$$\sigma_{PROC} >> \sigma_{WAFER} >> \sigma_{DIE}$$

$$\sigma_{DIE} >> \sigma_{LVAR}$$

$$\sigma_{DIE} >> \sigma_{|GRAD|}$$

σ_{LVAR} Strongly dependent upon area and layout

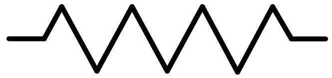
$$\sigma_{LVAR} \sim \frac{1}{\sqrt{\text{Area}}}$$

$$\sigma_{LVAR} \sim \text{Perimeter}$$

Relative size between σ_{LVAR} and $\sigma_{|GRAD|}$ dependent upon A, P, and process

Review from last lecture

Modeling process variations in semiconductor processes



R

- Statistics associated with value of dimensioned parameters (poles, GB, SR,R,C,transresistance gains, transconductance gains, … dominated by x_{RPROC})

- Statistics associated with matching/sensitive dimensionless parameters such as voltage or current gains, component ratios, pole Q, … (almost always closely placed) dominated by x_{RLGRAD} and x_{RLVAR} (because locally x_{RPROC}, x_{RWAFER}, x_{RDIE} are all correlated and equal)

- Gradients are dominantly linear if spacing is not too large

- Special layout techniques using common centroid approaches can be used to eliminate (or dramatically reduce) linear gradient effects so, if employed, matching/sensitive parameters dominated by x_{RLVAR} but occasionally common centroid layouts become impractical or areas become too large so that gradients become nonlinear and in these cases gradient effects will still limit performance

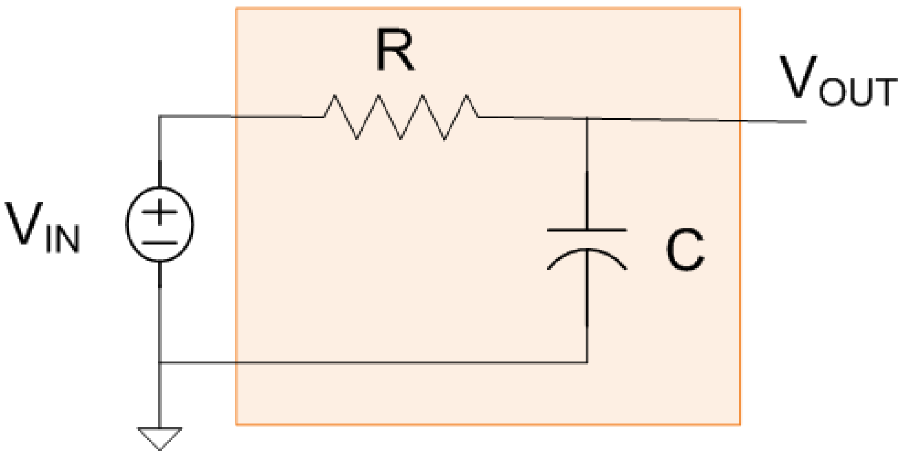
- Higher-order gradient effects can be eliminated with layout approaches that cancel higher "moments" but area and effort may not be attractive

Review from last lecture

Statistical Modeling of dimensioned parameters

Example:

Determine the standard deviation of the pole frequency (or band edge) of the first-order passive filter.



$$p = \frac{1}{RC}$$

Assume the process variables are zero mean Gaussian variable with standard deviations given by

$$\sigma_{\frac{R_{RPROC}}{R_{NOM}}} = 0.2 \qquad \sigma_{\frac{C_{RPROC}}{C_{NOM}}} = 0.1$$

Assume further that the effects of all other random components can be neglected

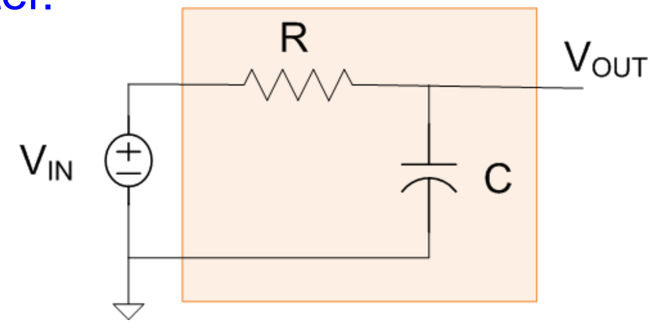
$$X = X_{NOM} + x_{RPROC} + \cancel{x_{RWAFER}} + \cancel{x_{RDIE}} + \cancel{x_{RLGRAD}} + \cancel{x_{RLVAR}}$$

Review from last lecture

Statistical Modeling of dimensioned parameters

Example (cont):

Determine the standard deviation of the pole frequency (or band edge) of the first-order passive filter.



$$p = \frac{1}{RC}$$

Assume the process variables are zero mean Gaussian variable with standard deviations given by

$$\sigma_{\frac{R_{RPROC}}{R_{NOM}}} = 0.2 \qquad \sigma_{\frac{C_{RPROC}}{C_{NOM}}} = 0.1$$

$$R = R_{NOM} + R_{PROC} \qquad C = C_{NOM} + C_{PROC}$$

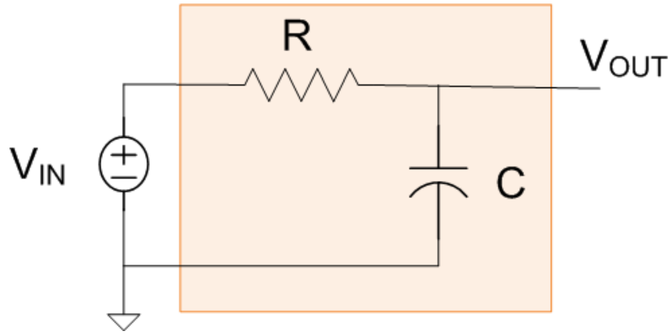
$$p = \frac{1}{(R_{NOM}+R_{PROC})(C_{NOM}+C_{PROC})} = \frac{1}{R_{NOM}C_{NOM} + R_{NOM}C_{PROC} + C_{NOM}R_{PROC} + R_{PROC}C_{PROC}}$$

- p is a multivariate random variable
- The pdf of p is extremely complicated

Review from last lecture

Example (cont):

Determine the standard deviation of the pole frequency (or band edge) of the first-order passive filter.



$$p = \frac{1}{RC}$$

Theorem: The sum of uncorrelated Gaussian random variables is a multivariate Gaussian random variable

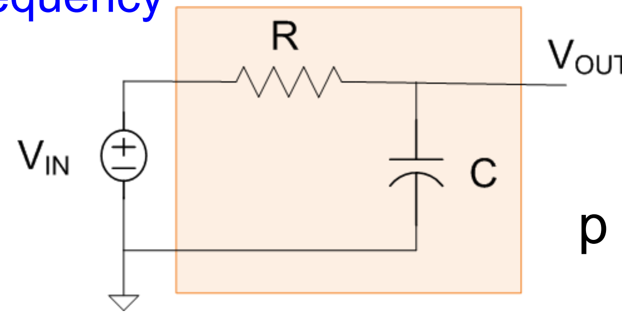
Theorem: If $X_1 \ldots X_m$ are uncorrelated random variables with standard deviations $\sigma_1, \sigma_2, \ldots \sigma_m$, and $a_1, a_2, \ldots a_m$ are constants, then the standard deviation of the random variable $y = \sum_{i=1}^{m} a_i X_i$ is given by the expression

$$\sigma_y = \sqrt{\sum_{i=1}^{m} a_i^2 \sigma_i^2}$$

Review from last lecture

Example (cont):

Determine the standard deviation of the pole frequency (or band edge) of the first-order passive filter.



$$p = \frac{1}{RC}$$

$$\sigma_{\frac{p}{p_{NOM}}} \simeq \sqrt{\sigma^2_{\frac{R_{RAN}}{R_{NOM}}} + \sigma^2_{\frac{C_{RAN}}{C_{NOM}}}}$$

But R_{RAN} and C_{RAN} are approximately R_{RPROC} and C_{RPROC}

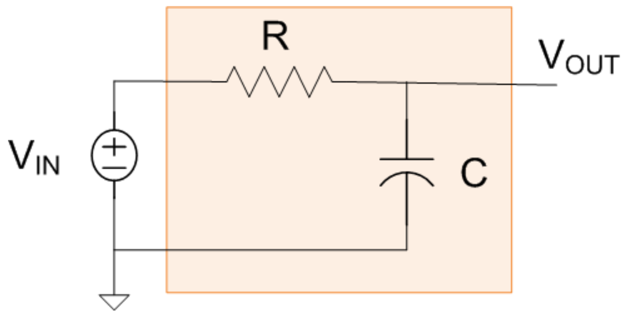
$$\sigma_{\frac{p}{p_{NOM}}} \simeq \sqrt{\sigma^2_{\frac{R_{RPROC}}{R_{NOM}}} + \sigma^2_{\frac{C_{RPROC}}{C_{NOM}}}}$$

recall

$$\sigma_{\frac{R_{RPROC}}{R_{NOM}}} = 0.2 \qquad \sigma_{\frac{C_{RPROC}}{C_{NOM}}} = 0.1$$

$$\sigma_{\frac{p}{p_{NOM}}} \simeq \sqrt{0.2^2 + 0.1^2} = 0.22$$

Review from last lecture



$$p = \frac{1}{RC}$$

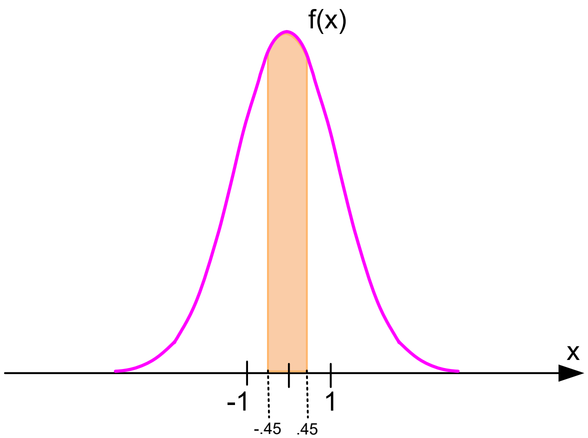
$$\sigma_{\frac{p}{p_{NOM}}} \simeq \sqrt{0.2^2 + 0.1^2} = 0.22$$

2. Determine the percent of the process lots that will have a pole with mean that is within 10% of the nominal value

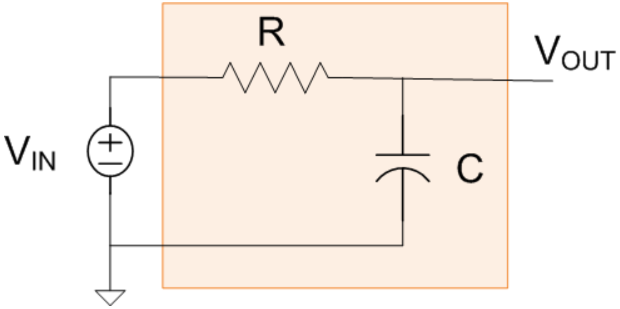
$$\theta_{prob} = 2F_{N(0,1)}(0.45) - 1$$

$$\theta_{prob} = 2 \bullet .6736 - 1 = 0.347$$

Thus, approximately 35% of the wafer lots will have a pole within 10% of the nominal value



Review from last lecture

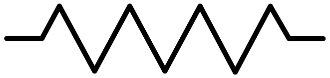


$$p = \frac{1}{RC}$$

$$\sigma_{\frac{p}{p_{NOM}}} \simeq \sqrt{0.2^2 + 0.1^2} = 0.22$$

3. What can the designer do to tighten the band edge of this filter?

Modeling process variations in semiconductor processes



R

- Most characteristics of interest in a filter (and many other circuits) are highly nonlinear functions of multiple random variables

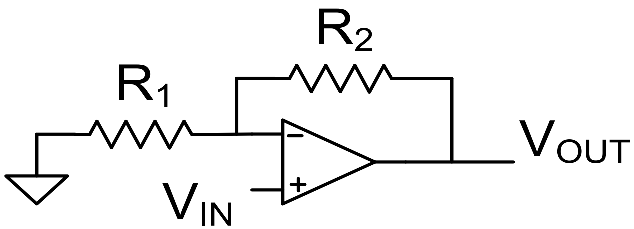
- Closed-form analytical expressions for pdf is often extremely difficult to obtain

- For most practical circuits, random component is small compared to the nominal component

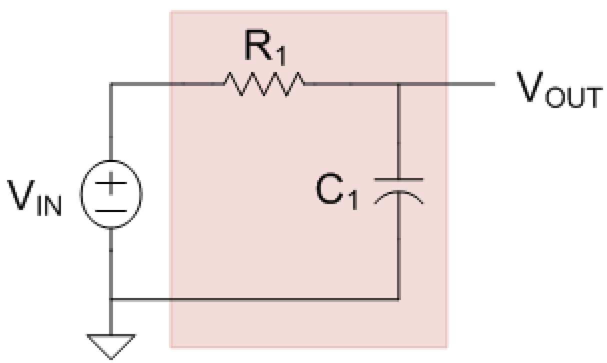
- Linearization of characteristics of interest for purpose of statistical analysis is usually quite accurate and drastically simplifies analysis

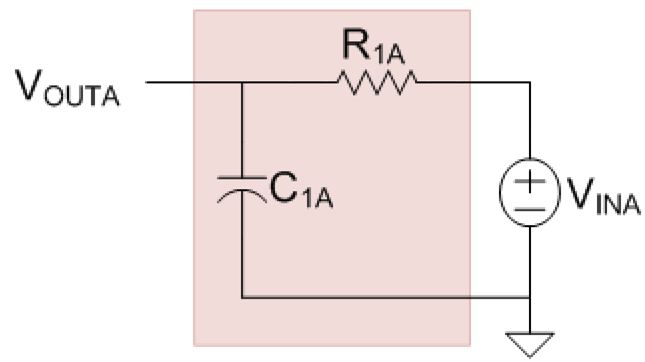
- Monte Carlo analysis is widely used for statistical characterization but is often very time consuming and gives little insight into design optimization

Statistical Modeling of Dimensionless Parameters



$$K = 1 + \frac{R_2}{R_1}$$





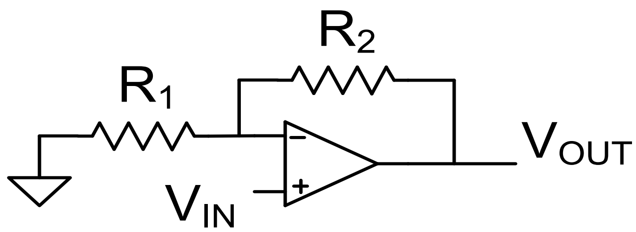
$$p_1 = \frac{1}{RC}$$

$$p_A = \frac{1}{R_{1A}C_{1A}}$$

$$\theta = \frac{p_A - p_1}{p_1}$$

Statistical Modeling of dimensionless parameters - example

Example 1



$$K = 1 + \frac{R_2}{R_1}$$

Determine the standard deviation of the voltage gain K

Determine the yield if the nominal gain is 10 $\pm 1\%$

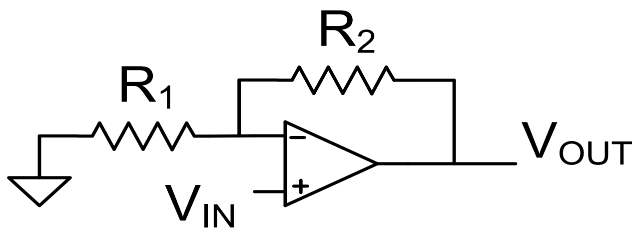
Assume a common centroid layout of R_1 and R_2 has been used and the area of R_1 is 100u^2 and both resistors have the same resistance density and R_2 is comprised of K-1 copies of R_1 . Neglect variable edge effects in the layout

Assume also that: A_ρ=.01µm $\qquad \sigma_{\frac{R_{PROC}}{R_{NOM}}} = 0.2$

A_ρ is the Pelgrom matching parameter

Statistical Modeling of dimensionless parameters - example

Example 1



$$K = 1 + \frac{R_2}{R_1}$$

Determine the standard deviation of the voltage gain K

$$K = 1 + \frac{R_{2N} + R_{2R}}{R_{1N} + R_{1R}}$$

$$K = 1 + \frac{R_{2N}\left(1 + \frac{R_{2R}}{R_{2N}}\right)}{R_{1N}\left(1 + \frac{R_{1R}}{R_{1N}}\right)}$$

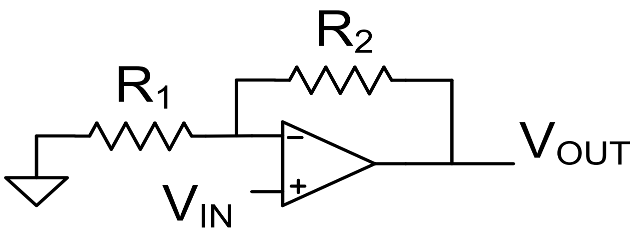
$$K \cong 1 + \frac{R_{2N}}{R_{1N}}\left(1 + \frac{R_{2R}}{R_{2N}}\right)\left(1 - \frac{R_{1R}}{R_{1N}}\right)$$

$$K \cong 1 + \frac{R_{2N}}{R_{1N}}\left(1 + \frac{R_{2R}}{R_{2N}} - \frac{R_{1R}}{R_{1N}}\right)$$

$$K \cong \left(1 + \frac{R_{2N}}{R_{1N}}\right) + \frac{R_{2N}}{R_{1N}}\left(\frac{R_{2R}}{R_{2N}} - \frac{R_{1R}}{R_{1N}}\right)$$

Statistical Modeling of dimensionless parameters - example

Example 1



$$K = 1 + \frac{R_2}{R_1}$$

Determine the standard deviation of the voltage gain K

$$K \cong \left(1 + \frac{R_{2N}}{R_{1N}}\right) + \frac{R_{2N}}{R_{1N}}\left(\frac{R_{2R}}{R_{2N}} - \frac{R_{1R}}{R_{1N}}\right)$$

$$K \cong K_N + (K_N - 1)\left(\frac{R_{2R}}{R_{2N}} - \frac{R_{1R}}{R_{1N}}\right)$$

$$R_{2R} \cong R_{2RPROC} + R_{2RGRAD} + R_{2RLVAR}$$

$$R_{1R} \cong R_{1RPROC} + R_{1RGRAD} + R_{1RLVAR}$$

$$K \cong K_N + (K_N - 1)\left(\frac{R_{2RPROC} + R_{2RGRAD} + R_{2RLVAR}}{R_{2N}} - \frac{R_{1RPROC} + R_{1RGRAD} + R_{1RLVAR}}{R_{1N}}\right)$$

But R_{2RPROC} and R_{1RPROC} are correlated

$$R_{2RPROC} = (K_N - 1) R_{1RPROC}$$

And, since a common centroid layout is used,

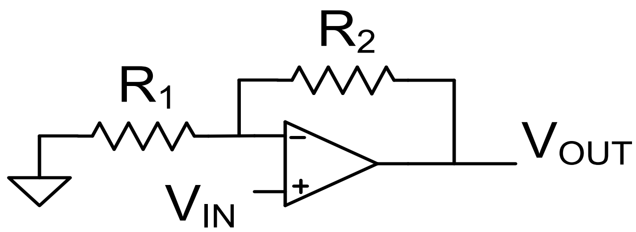
R_{2RGRAD} and R_{1RGRAD} are correlated

$$R_{2RGRAD} = (K_N - 1) R_{1RGRAD}$$

R_{2RLVAR} and R_{1RLVAR} are uncorrelated

Statistical Modeling of dimensionless parameters - example

Example 1



$$K = 1 + \frac{R_2}{R_1}$$

Determine the standard deviation of the voltage gain K

$$K \cong K_N + (K_N - 1)\left(\frac{R_{2RPROC} + R_{2RGRAD} + R_{2RLVAR}}{R_{2N}} - \frac{R_{2RPROC} + R_{2RGRAD} + R_{2RLVAR}}{R_{1N}}\right)$$

$$K \cong K_N + (K_N - 1)\left(\frac{(K_N - 1)R_{1RPROC} + (K_N - 1)R_{1RGRAD} + R_{2RLVAR}}{R_{2N}} - \frac{R_{1RPROC} + R_{1RGRAD} + R_{1RLVAR}}{R_{1N}}\right)$$

Since $R_{2N}=(K_N-1)R_{1N}$

$$K \cong K_N + (K_N - 1)\left(\frac{(K_N - 1)R_{1RPROC} + (K_N - 1)R_{1RGRAD}}{(K_N - 1)R_{1N}} + \frac{R_{2RLVAR}}{R_{2N}} - \frac{R_{1RPROC} + R_{1RGRAD} + R_{1RLVAR}}{R_{1N}}\right)$$

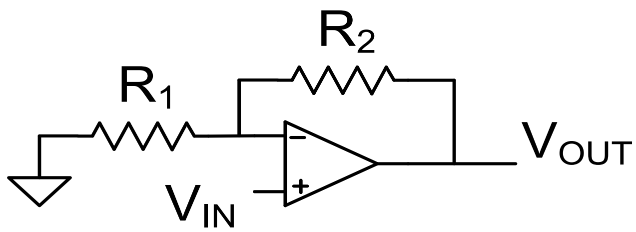
$$K \cong K_N + (K_N - 1)\left(\left[\frac{(K_N - 1)R_{1RPROC} + (K_N - 1)R_{1RGRAD}}{(K_N - 1)R_{1N}} - \frac{R_{1RPROC} + R_{1RGRAD}}{R_{1N}}\right] + \frac{R_{2RLVAR}}{R_{2N}} - \frac{R_{1RLVAR}}{R_{1N}}\right)$$

$$K \cong K_N + (K_N - 1)\left(\frac{R_{2RLVAR}}{R_{2N}} - \frac{R_{1RLVAR}}{R_{1N}}\right)$$

K not dependent on R_{RPROC} !!

Statistical Modeling of dimensionless parameters - example

Example 1



$$K = 1+\frac{R_2}{R_1}$$

Determine the standard deviation of the voltage gain K

$$K \cong K_N + (K_N - 1)\left(\frac{R_{2RLVAR}}{R_{2N}} - \frac{R_{1RLVAR}}{R_{1N}}\right)$$

Recall: $$\sigma_{\frac{p}{p_{NOM}}} \cong \sqrt{0.2^2 + 0.1^2} = 0.22$$ (p was the pole of a dimensioned parameter)

$$\sigma_{\frac{K}{K_N}} \cong \left(1 - \frac{1}{K_N}\right)\sqrt{\sigma^2_{\frac{R_{2R}}{R_{2N}}} + \sigma^2_{\frac{R_{1R}}{R_{1N}}}}$$

Statistical characterization of local random variations of resistors and capacitors

Theorem: If the perimeter variations and contact resistance are neglected, the standard deviation of the local random variations of a resistor of area A is given by the expression

$$\sigma_{\frac{R}{R_N}} = \frac{A_\rho}{\sqrt{A}}$$

A_ρ is a constant (has dimensions of μm) and is not related to area!

Theorem: If the perimeter variations are neglected, the standard deviation of the local random variations of a capacitor of area A is given by the expression

$$\sigma_{\frac{C}{C_N}} = \frac{A_C}{\sqrt{A}}$$

A_C is a constant (has dimensions of μm) and is not related to area!

Note both of these expressions are independent of the value of R and C

Statistical characterization of local random variations of MOS transistor parameters

Theorem: If the perimeter variations are neglected, the variance of the local random variations of the normalized threshold voltage of a rectangular MOS transistor of dimensions W and L is given by the expression

$$\sigma^2_{\frac{V_T}{V_{T_N}}} = \frac{A^2_{VTO}}{V^2_{T_N} WL} \qquad \text{or as} \qquad \sigma^2_{\frac{V_T}{V_{T_N}}} = \frac{A^2_{VT}}{WL}$$

Theorem: If the perimeter variations are neglected, the variance of the local random variations of the normalized C_{OX} of a rectangular MOS transistor of dimensions W and L is given by the expression

$$\sigma^2_{\frac{C_{OX}}{C_{OXN}}} = \frac{A^2_{COX}}{WL}$$

Theorem: If the perimeter variations are neglected, the variance of the local random variations of the normalized mobility of a rectangular MOS transistor of dimensions W and L is given by the expression

$$\sigma^2_{\frac{\mu_R}{\mu_N}} = \frac{A^2_{\mu}}{WL}$$

where the parameters A_X are all constants characteristic of the process (i.e. model parameters)

Statistical characterization of local random variations of MOS transistor parameters

$$\sigma_{\frac{R}{R_N}} = \frac{A_\rho}{\sqrt{A}} \qquad \sigma_{\frac{C}{C_N}} = \frac{A_C}{\sqrt{A}}$$

$$\sigma^2_{\frac{C_{OX}}{C_{OXN}}} = \frac{A^2_{COX}}{WL} \qquad \sigma^2_{\frac{\mu_R}{\mu_N}} = \frac{A^2_{\mu}}{WL} \qquad \sigma^2_{\frac{V_T}{V_{T_N}}} = \frac{A^2_{VTO}}{V^2_{T_N} WL}$$

- The parameters A_ρ, A_C, A_μ, A_{COX}, and A_{VT0} are often termed “Pelgrom” parameters and are part of the PDK of a process

Matching properties of MOS transistors [PDF

MJM Pelgrom, ACJ Duinmaijer… - IEEE Journal of solid …, 1989 - ieeexplore.ieee.org

The **matching properties** of the threshold voltage, substrate factor, and current factor of **MOS transistors** have been analyzed and measured. Improvements to the existing theory are …

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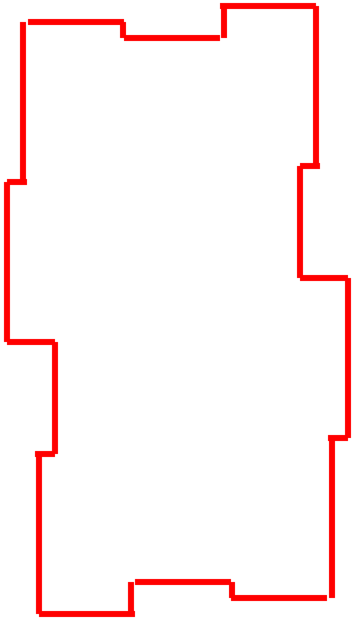
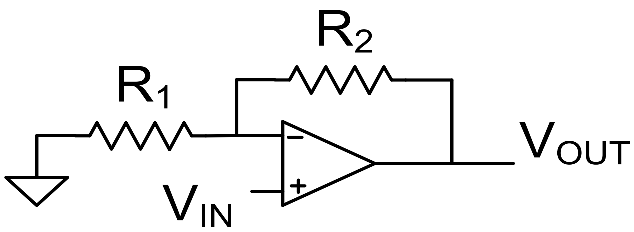
- The effects of edge variations (but not roughness) on the variance of resistors, capacitors, and transistors can readily be included (though not precisely modeled) but for most layouts is dominated by the area dependent variations
- There is some correlation between the model parameters of MOS transistors but they are often ignored to simplify calculations



Can’t precisely model this type of W and L variations. Must maintain no variations on any of the four boundaries though the boundaries themselves can be modeled as random variables.

Statistical Modeling of dimensionless parameters - example

Example 1



$$K = 1 + \frac{R_2}{R_1}$$

Determine the standard deviation of the voltage gain K

$$K \cong K_N + (K_N - 1)\left(\frac{R_{2RLVAR}}{R_{2N}} - \frac{R_{1RLVAR}}{R_{1N}}\right)$$

$$\sigma_K \cong (K_N - 1)\sqrt{\sigma^2_{\frac{R_{2R}}{R_{2N}}} + \sigma^2_{\frac{R_{1R}}{R_{1N}}}}$$

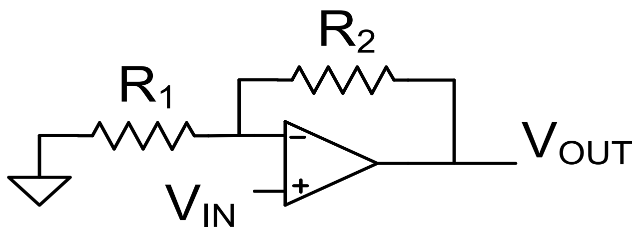
$$\sigma_{\frac{R}{R_N}} = \frac{A_\rho}{\sqrt{A}}$$

$$\sigma_K \cong (K_N - 1)A_\rho\sqrt{\frac{1}{A_{R2}} + \frac{1}{A_{R1}}}$$

$$\sigma_K \cong (K_N - 1)A_\rho\sqrt{\frac{1}{(K_N - 1)A_{R1}} + \frac{1}{A_{R1}}}$$

Statistical Modeling of dimensionless parameters - example

Example 1



$$K = 1 + \frac{R_2}{R_1}$$

Determine the standard deviation of the voltage gain K

$$\sigma_K \cong (K_N - 1) A_\rho \sqrt{\frac{1}{(K_N - 1) A_{R1}} + \frac{1}{A_{R1}}}$$

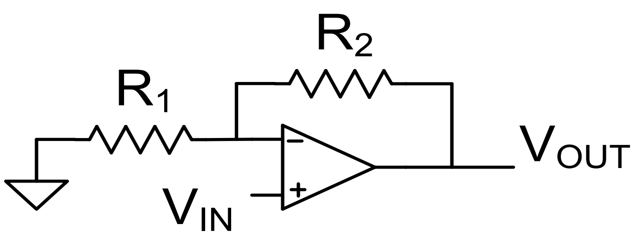
$$\sigma_K \cong (K_N - 1) \frac{A_\rho}{\sqrt{A_{R1}}} \sqrt{\frac{1}{(K_N - 1)} + 1} = \frac{A_\rho}{\sqrt{A_{R1}}} (K_N - 1) \sqrt{\frac{K_N}{(K_N - 1)}}$$

$$\sigma_K \cong \frac{A_\rho}{\sqrt{A_{R1}}} \sqrt{K_N (K_N - 1)}$$

$$\sigma_{\frac{K}{K_N}} \cong \frac{A_\rho}{\sqrt{A_{R1}}} \sqrt{1 - \frac{1}{K_N}}$$

Statistical Modeling of dimensionless parameters - example

Example 1



$$K = 1+\frac{R_2}{R_1}$$

Determine the standard deviation of the voltage gain K

$$\sigma_K \cong \frac{A_\rho}{\sqrt{A_{R1}}}\sqrt{K_N\left(K_N-1\right)}$$

A_ρ=.01u $\quad$ A_{R1}=100u^2 $\quad$ $\sigma_{\frac{R_{PROC}}{R_{NOM}}} = 0.2$

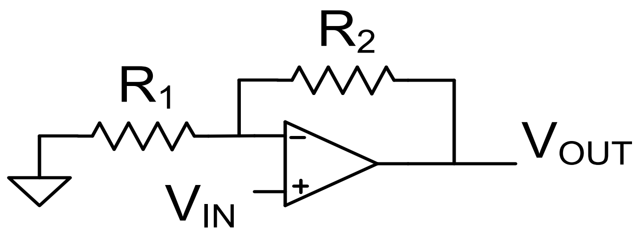
$$\sigma_K \cong \frac{.01}{10}\sqrt{K_N\left(K_N-1\right)} = .001\sqrt{K_N\left(K_N-1\right)}$$

$$\sigma_{\frac{K}{K_N}} \cong .001\sqrt{1-\frac{1}{K_N}}$$

- The standard deviation can be improved by increasing area but a 4X increase in area is needed for a 2X reduction in sigma
- Note the standard deviation of the normalized gain is much smaller than the standard deviation of the process variations

Statistical Modeling of dimensionless parameters - example

Example 1



$$K = 1 + \frac{R_2}{R_1}$$

Determine the standard deviation of the voltage gain K

$$\sigma_{\frac{K}{K_N}} \cong .001\sqrt{1 - \frac{1}{K_N}}$$

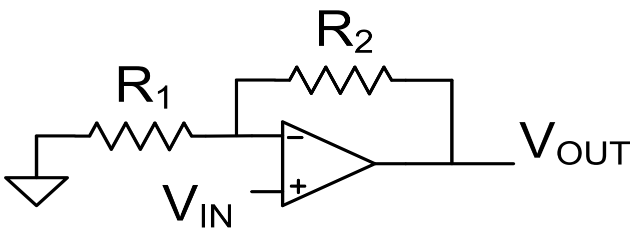
Determine the yield if the nominal gain is 10 $\pm 1\%$

$$\sigma_{\frac{K}{K_N}} \cong .001\sqrt{1 - \frac{1}{10}} = .00095$$

$$\frac{K}{K_N} \cong N(1,\ 0.00095)$$

Statistical Modeling of dimensionless parameters - example

Example 1



$$K = 1 + \frac{R_2}{R_1}$$

Determine the yield if the nominal gain is 10 $\pm 1\%$

$$\frac{K}{K_N} \cong N(1,\ 0.00095)$$

$$\frac{\frac{K}{K_N} - 1}{0.00095} \cong N(0,1)$$

$$9.9 < K < 10.1$$

$$.99 < \frac{K}{K_N} < 1.01$$

$$-.01 < \frac{K}{K_N} - 1 < .01$$

$$-10 < \frac{\frac{K}{K_N} - 1}{.00095} < 10$$

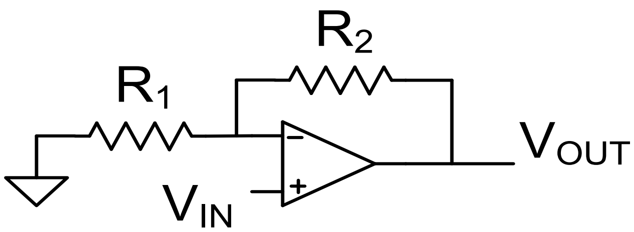
These are 10 sigma values !

The gain yield is essentially 100%

Could substantially decrease area or increase gain accuracy if desired

Statistical Modeling of dimensionless parameters - example

Example 1



$$K = 1 + \frac{R_2}{R_1}$$

Determine the yield if the nominal gain is 10 $\pm 1\%$

$$-10 < \frac{\frac{K}{K_N} - 1}{.00095} < 10$$

These are 10 sigma values !

The gain yield is essentially 100%

What effect did the very large value of the process variance have on yield?

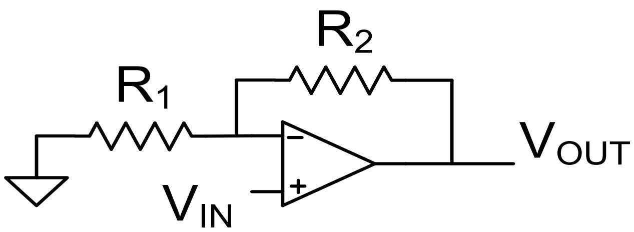
A_ρ=.01μm

$$\sigma_{\frac{R_{PROC}}{R_{NOM}}} = 0.2$$

No effect !! This dimensionless parameter not dependent on process variations

Statistical Modeling of dimensionless parameters - example

Example 2



$$K = 1 + \frac{R_2}{R_1}$$

Determine the yield if the gain is to be 10 $\pm 1\%$

Assume a common centroid layout of R_1 and R_2 has been used and the area of R_1 is 10u^2 and both resistors have the same resistance density and R_2 is comprised of K-1 copies of R_1 . Neglect variable edge effects in the layout

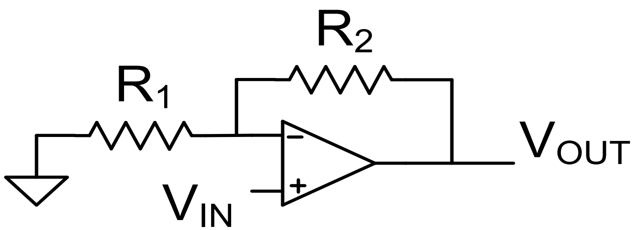
A_ρ=.025µm

$$\sigma_{\frac{R_{PROC}}{R_{NOM}}} = 0.2$$

Note this is simply a 10X reduction in area from previous example and an increase in A_ρ by a factor of 2.5

Statistical Modeling of dimensionless parameters - example

Example 2



$$K = 1+\frac{R_2}{R_1}$$

Determine the standard deviation of the voltage gain K

$$\sigma_K \cong \frac{A_\rho}{\sqrt{A_{R1}}}\sqrt{K_N\left(K_N-1\right)}$$

A_ρ=.025um A_{R1}=10um^2

$$\sigma_{\frac{R_{PROC}}{R_{NOM}}} = 0.2$$

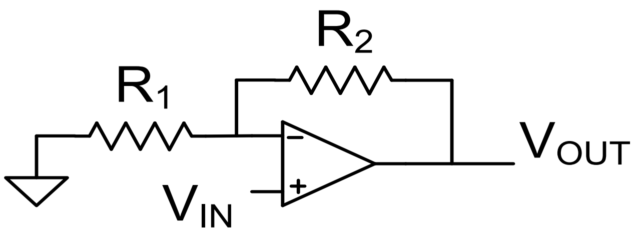
$$\sigma_K \cong \frac{.025}{\sqrt{10}}\sqrt{K_N\left(K_N-1\right)} = .0079\sqrt{K_N\left(K_N-1\right)}$$

$$\sigma_{\frac{K}{K_N}} \cong .0079\sqrt{1-\frac{1}{K_N}}$$

Note the standard deviation of the normalized gain is independent of the very large $\sigma_{\frac{R_{PROC}}{R_{NOM}}}$

Statistical Modeling of dimensionless parameters - example

Example 2



$$K = 1 + \frac{R_2}{R_1}$$

Determine the standard deviation of the voltage gain K

$$\sigma_{\frac{K}{K_N}} \cong .0079\sqrt{1-\frac{1}{K_N}}$$

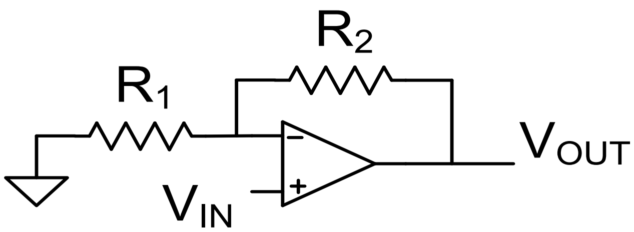
Determine the yield if the gain is to be 10 $\pm 1\%$

$$\sigma_{\frac{K}{K_N}} \cong .0079\sqrt{1-\frac{1}{10}} = .0075$$

$$\frac{K}{K_N} \cong N(1,\ 0.0075)$$

Statistical Modeling of dimensionless parameters - example

Example 2



$$K = 1 + \frac{R_2}{R_1}$$

Determine the yield if the nominal gain is 10 $\pm 1\%$

$$\frac{K}{K_N} \cong N(1,\ 0.0075)$$

$$\frac{\frac{K}{K_N} - 1}{0.0075} \cong N(0,1)$$

$$9.9 < K < 10.1$$

$$.99 < \frac{K}{K_N} < 1.01$$

$$-.01 < \frac{K}{K_N} - 1 < .01$$

$$-1.33 < \frac{\frac{K}{K_N} - 1}{.0075} < 1.33$$

Have dropped from 10 sigma to 1.33 sigma boundaries

$$Y = 2F_{N(0,1)}(1.33) - 1 = 2^*.9082 - 1 = 0.8164$$

Dramatic drop from 100% yield to about 82% yield!

Statistical Modeling of Filter Characteristics

The variance of dimensioned filter parameters (e.g. ω_0, poles, band edges, …) is often very large due to the process-level random variables which dominate

The variance of dimensionless filter parameters (e.g. Q, gain, …) are often quite small since in a good design they will depend dominantly on local random variations which are much smaller than process-level variations

The variance of dimensionless filter parameters is invariably proportional to the reciprocal of the square root of the relevant area and thus can be managed with appropriate area allocation

Linearization of Functions of a Random Variable

- Characteristics of most circuits of interest are themselves random variables
- Relationship between characteristics and the random variables often highly nonlinear
- Ad Hoc manipulations (repeated Taylor's series expansions) were used to linearize the characteristics in terms of the random variables

$$Y \cong Y_N + \sum_{i=1}^{n} \left(a_i x_{Ri} \right)$$

- This is important because if the random variables are uncorrelated the variance of the characteristic can be readily obtained

$$\sigma_Y^2 \cong \sum_{i=1}^{n} \left(a_i^2 \sigma_{x_{Ri}}^2 \right)$$

$$\sigma_{\frac{Y}{Y_N}}^2 \cong \frac{1}{Y_N^2} \bullet \sum_{i=1}^{n} \left(a_i^2 \sigma_{x_{Ri}}^2 \right)$$

- This approach was applicable since the random variables are small
- These Ad Hoc manipulations can be formalized and this follows

Formalization of Statistical Analysis

Consider a function of interest Y

$$Y = f\left(x_{1N}, x_{2N}, \ldots x_{nN}, : x_{1R}, x_{2R}, \ldots x_{nR}\right) = f\left(\left[X_N\right], \left[X_R\right]\right)$$

This can be expressed in a multi-variate power series as

$$Y \cong f\left(\left[X_N\right] \ , \left[X_R\right]\right)\Big|_{\left[X_R\right]=[0]} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\bigg|_{\left[X_N\right],\left[X_R\right]=[0]} \bullet x_{Ri}\right) + \sum_{\substack{i=1 \\ j=1}}^{\substack{n \\ n}}\left(\frac{\partial^2 f}{\partial x_j \partial x_i}\bigg|_{\left[X_N\right],\left[X_R\right]=[0]} \bullet x_{Ri} x_{Rj}\right) + \ldots.$$

If the random variables are small compared to the nominal variables

$$Y \cong f\left(\left[X_N\right] \ , \left[X_R\right]\right)\Big|_{\left[X_R\right]=[0]} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\bigg|_{\left[X_N\right],\left[X_R\right]=[0]} \bullet x_{Ri}\right)$$

If the random variable are uncorrelated, it follows that

$$\sigma_Y^2 = \sum_{i=1}^{n}\left(\left[\frac{\partial f}{\partial x_i}\bigg|_{\left[X_N\right],\left[X_R\right]=[0]}\right]^2 \bullet \sigma_{x_{Ri}}^2\right)$$

$$\sigma_{\frac{Y}{Y_N}}^2 = \frac{1}{Y_N^2}\sum_{i=1}^{n}\left(\left[\frac{\partial f}{\partial x_i}\bigg|_{\left[X_N\right],\left[X_R\right]=[0]}\right]^2 \bullet \sigma_{x_{Ri}}^2\right)$$

Formalization of Statistical Analysis

$$Y = f\left(x_{1N}, x_{2N}, \ldots x_{nN}, : x_{1R}, x_{2R}, \ldots x_{nR}\right) = f\left([X_N],[X_R]\right)$$

$$\sigma^2_{\frac{Y}{Y_N}} = \frac{1}{Y_N^2} \sum_{i=1}^{n} \left(\left[\left. \frac{\partial f}{\partial x_i} \right|_{[X_N],[X_R]=[0]} \right]^2 \bullet \sigma^2_{x_{Ri}} \right)$$

Recall:

$$S_x^f = \frac{\partial f}{\partial x} \frac{x}{f} \quad \Rightarrow \quad \left. \left(\frac{\partial f}{\partial x_i} \right)^2 \right|_{[X_N],[x_R]=0} = \left. \left(S_{x_i}^f \right)^2 \right|_{[X_N],[x_R]=0} \bullet \frac{Y_N{}^2}{X_{N_i}^2}$$

Y=f

Thus:

$$\sigma^2_{\frac{Y}{Y_N}} = \sum_{i=1}^{n} \left(\left[\left. S_{x_i}^f \right|_{[X_N]} \right]^2 \bullet \sigma^2_{\frac{x_{Ri}}{X_{Ni}}} \right)$$

- Sensitivity analysis often used for statistical characterization of filter performance
- This is often much faster and less tedious than doing the linearization as described above though actually concepts are identical

Stay Safe and Stay Healthy !

End of Lecture 14